

## Wilson loops as $D3$ -branes

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**ABSTRACT:** We prove that the half-BPS Wilson loop operator of  $\mathcal{N} = 4$  SYM in a symmetric representation of the gauge group has a bulk gravitational description in terms of a single  $D3$ -brane in  $\text{AdS}_5 \times S^5$ , as argued in hep-th/0604007. We also show that a half-BPS Wilson loop operator in an arbitrary representation is described by the  $D3$ -brane configuration proposed in hep-th/0604007. This is demonstrated by explicitly integrating out the degrees of freedom on the  $D3$ -branes and showing that they insert a half-BPS Wilson loop operator into the  $\mathcal{N} = 4$  SYM path integral in the desired representation.

**KEYWORDS:** Gauge-gravity correspondence, Brane Dynamics in Gauge Theories.

**Contents**

<b>1.</b>	<b>Introduction</b>	<b>1</b>
<b>2.</b>	<b>A <math>D3_k</math>-brane as a Wilson loop in the <math>k</math>-th symmetric representation</b>	<b>2</b>
<b>3.</b>	<b>Multiple <math>D3_k</math>-branes as Wilson loop in arbitrary representation</b>	<b>5</b>
<b>A.</b>	<b>Gauge theory along Coulomb branch</b>	<b>8</b>

**1. Introduction**

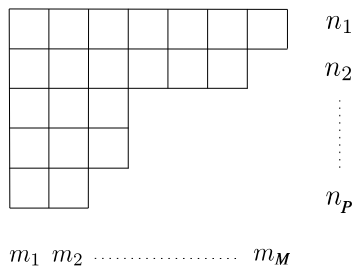
Recently, the bulk description of all half-BPS Wilson loop operators of  $\mathcal{N} = 4$  SYM has been given [1] in terms of D-branes in  $\text{AdS}_5 \times \text{S}^5$  (see [2, 3] for previous work). A half-BPS Wilson loop — labeled by a representation of  $\text{U}(N)$  with Young tableau — can be described [1] in terms of  $M$   $D5$ -branes or alternatively in terms of  $P$   $D3$ -branes in  $\text{AdS}_5 \times \text{S}^5$ . This generalizes the bulk description of a Wilson loop in the fundamental representation in terms of a string worldsheet [4, 5] to all other representations. For other work see e.g. [6–12].

In [1], it was argued that the  $D3_k$ -brane<sup>1</sup> solution in  $\text{AdS}_5 \times \text{S}^5$  of [4, 2] corresponds to a half-BPS Wilson loop operator in the  $k$ -th symmetric representation, while a Wilson loop operator in the representation given by figure 1 corresponds to the array of  $P$  branes  $(D3_{n_1}, D3_{n_2}, \dots, D3_{n_P})$ .

In this note we give a first principles derivation of this proposal. We complete the analysis in [1] by studying the flat space brane configuration which yields in the near horizon limit the  $P$   $D3$ -branes  $(D3_{n_1}, D3_{n_2}, \dots, D3_{n_P})$  in  $\text{AdS}_5 \times \text{S}^5$  dual to the Wilson loop operator labeled by figure 1. This configuration corresponds to separating  $P$   $D$ -branes by a distance  $L$  from a stack of  $N + P$  coincident  $D3$ -branes and introducing  $k$  fundamental strings stretched between the two stacks of branes, in the limit  $L \rightarrow \infty$ . We can exactly integrate out the degrees of freedom introduced by the extra  $P$   $D$ -branes from the low energy effective field theory describing this configuration and show that the net effect is to insert into the  $\text{U}(N)$   $\mathcal{N} = 4$  SYM path integral a Wilson loop operator with the expected representation, thus explicitly confirming the proposal in [1].

The plan of the rest of this note is as follows. In section 2 we show that a single  $D3$ -brane in  $\text{AdS}_5 \times \text{S}^5$  with  $k$  units of fundamental string charge corresponds to a half-BPS Wilson loop in the  $k$ -th symmetric representation of  $\text{U}(N)$ . This is shown by studying in a certain infinite mass limit the Coulomb branch of  $\mathcal{N} = 4$  SYM in the presence of  $k$  W-bosons. In section 3 this result is generalized to arbitrary representations and reproduces the proposal in [1]. Some details of the infinite mass limit of the Coulomb branch of  $\mathcal{N} = 4$  SYM are relegated to the appendix.

<sup>1</sup> $k$  is the fundamental string charge dissolved on the brane.



**Figure 1:** A Young tableau. For  $U(N)$   $P \leq N$  and  $M$  is arbitrary.

## 2. A $D3_k$ -brane as a Wilson loop in the $k$ -th symmetric representation

In [1], it was argued that the  $D3_k$ -brane solution in  $AdS_5 \times S^5$  of [4, 2] corresponds to a half-BPS Wilson loop operator labeled by the following Young tableau:

$$\boxed{1 \ 2 \ \cdot \ \cdot \ \cdot \ \cdot \ k} \quad (2.1)$$

This solution [4, 2] has an  $AdS_2 \times S^2$  worldvolume geometry and carries  $k$  units of fundamental string charge. The fact that  $k$  is arbitrary, that there can be at most  $N$  such  $D3$ -branes in  $AdS_5 \times S^5$ , and its proposed relation through bosonization to the defect conformal field theory derived for the  $D5_k$ -brane<sup>2</sup> led [1] to the abovementioned proposal.

In this note we show that this proposal is indeed correct by studying a brane configuration in flat space. We integrate out the physics on the brane and show that the  $D$ -brane inserts the desired Wilson loop into the  $\mathcal{N} = 4$  SYM path integral. This brane configuration can also be studied in the near horizon limit and indeed reproduces the  $D3_k$ -brane solution of [4, 2].

A half-BPS Wilson loop of  $\mathcal{N} = 4$  SYM in a representation<sup>3</sup>  $R$

$$W_R = \text{Tr}_R P \exp \left( i \int dt (A_0 + \phi) \right), \quad (2.2)$$

is obtained by adding a static, infinitely massive charged probe to  $\mathcal{N} = 4$  SYM. As already shown in [4, 5] (see also [13]), one way of introducing external charges in  $U(N)$   $\mathcal{N} = 4$  SYM is to consider a stack of  $N + 1$   $D3$ -branes and going along the Coulomb brach of the gauge theory.

Let's consider the gauge theory on  $N + 1$   $D3$ -branes and break the gauge symmetry down to  $U(N) \times U(1)$  by separating one of the branes. In the gauge theory description this

<sup>2</sup>This  $D5$ -brane, which has an  $AdS_2 \times S^4$  worldvolume geometry and  $k \leq N$  units of fundamental string charge, was shown to correspond to a Wilson loop in the  $k$ -th antisymmetric representation — a Young tableau with  $k$  boxes in one column — by integrating out the degrees of freedom on the  $D5$ -brane.

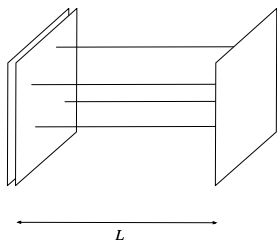
<sup>3</sup> $R = (R_1, R_2, \dots, R_N)$ , with  $R_i \geq R_{i+1}$  labels a representation of  $U(N)$  given by a Young tableau with  $R_i$  boxes in the  $i$ -th row.

corresponds to turning on the following expectation value

$$\langle \phi \rangle = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}, \tag{2.3}$$

where  $\phi$  is one of the scalar fields of  $\mathcal{N} = 4$  SYM, thus breaking the  $SO(6)$  R-symmetry of  $\mathcal{N} = 4$  SYM down to  $SO(5)$ .

We are interested in studying the low energy physics of this D-brane configuration in a background where  $k$  static fundamental strings are stretched between the two stacks of  $D3$ -branes:



**Figure 2:** Two separated stacks of  $D3$ -branes with  $k$  fundamental strings stretched between them.

In the gauge theory description, we must study the low energy effective field theory of  $U(N + 1)$   $\mathcal{N} = 4$  SYM when spontaneously broken to  $U(N) \times U(1)$ . The presence of  $k$  stretched static fundamental strings corresponds to inserting at  $t \rightarrow -\infty$   $k$  W-boson creation operators  $w^\dagger$  and  $k$  W-boson annihilation operators  $w$  at  $t \rightarrow \infty$ . Since we are interested in the limit when the charges are infinitely massive probes, we must study this field theory vacuum in the limit  $L \rightarrow \infty$ . In this limit the  $U(1)$  theory completely decouples from the  $U(N)$  theory.

Physically, the  $L \rightarrow \infty$  limit can be thought of as a non-relativistic limit. The dynamics can be conveniently extracted by defining

$$w = \frac{1}{\sqrt{L}} e^{-itL} \chi, \tag{2.4}$$

making the kinetic term for the W-bosons non-relativistic. As shown in the appendix, the terms in the effective action surviving the limit are given by

$$S = S_{\mathcal{N}=4} + S_\chi, \tag{2.5}$$

where:

$$S_\chi = \int i\chi^\dagger \partial_t \chi + \chi^\dagger (A_0 + \phi) \chi. \tag{2.6}$$

Therefore, the path integral describing  $k$  fundamental strings stretching between the two stacks of  $D$ -branes in the  $L \rightarrow \infty$  limit is given by<sup>4</sup>

$$Z \equiv e^{iS_{\mathcal{N}=4}} \int [D\chi][D\chi^\dagger] e^{iS_\chi} \frac{1}{k!} \sum_{i_1, \dots, i_k} \chi_{i_1}(\infty) \chi_{i_2}(\infty) \dots \chi_{i_k}(\infty) \chi_{i_1}^\dagger(-\infty) \chi_{i_2}^\dagger(-\infty) \dots \chi_{i_k}^\dagger(-\infty), \tag{2.7}$$

where  $i_l = 1, \dots, N$  is a fundamental index of  $U(N)$ .

From the formula for the W-boson propagator that follows from (2.6)<sup>5</sup>

$$\langle \chi_i(t_1) \chi_j^\dagger(t_2) \rangle = \theta(t_1 - t_2) \delta_{ij}, \tag{2.8}$$

<sup>4</sup>The path integral over the  $U(N)$   $\mathcal{N} = 4$  SYM is to be performed at the end.

<sup>5</sup> $\theta(t)$  is the Heaviside step function.

one can derive the following “effective” propagator

$$\langle \chi_i(\infty) \chi_j^\dagger(-\infty) \rangle_{\text{eff}} \equiv \langle \exp \left( i \int dt i \chi^\dagger \partial_t \chi + \chi^\dagger (A_0 + \phi) \chi \right) \chi_i(\infty) \chi_j^\dagger(-\infty) \rangle = U_{ij}, \quad (2.9)$$

where  $U$  is the holonomy matrix appearing in the half-BPS Wilson loop operator (2.2):

$$U = P \exp \left( i \int dt (A_0 + \phi) \right) \in \text{U}(N). \quad (2.10)$$

Using this “effective” propagator we can now evaluate (2.7). We must sum over all Wick contractions between the  $W$ -bosons. Contractions are labeled by a permutation  $\omega$  of the symmetric group  $S_k$ . The path integral (2.7) is then given by:

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \frac{1}{k!} \sum_{\omega \in S_k} U_{i_{\omega(1)}}^{i_1} \cdots U_{i_{\omega(k)}}^{i_k}. \quad (2.11)$$

Permutations having the same cycle structure upon decomposing a permutation into the product of disjoint cycles give identical contributions in (2.11). Since all elements in a given conjugacy class of  $S_k$  have the same cycle structure, we can replace the sum over permutations  $\omega$  in (2.11) by a sum over conjugacy classes  $C(\vec{k})$  of  $S_k$ . Conjugacy classes of  $S_k$  are labeled by partitions of  $k$ , denoted by  $\vec{k}$ , so that

$$k = \sum_{l=1}^k l k_l, \quad (2.12)$$

and each permutation in the conjugacy class has  $k_l$  cycles of length  $l$ .

Therefore, (2.11) can be written as

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \frac{1}{k!} \sum_{C(\vec{k})} N_{C(\vec{k})} \gamma_{\vec{k}}(U), \quad (2.13)$$

where

$$\gamma_{\vec{k}}(U) = \prod_{l=1}^k (\text{Tr} U^l)^{k_l}, \quad (2.14)$$

and  $N_{C(\vec{k})}$  is the number of permutations in the conjugacy class  $C(\vec{k})$ , which is given by

$$N_{C(\vec{k})} = \frac{k!}{z_{\vec{k}}}, \quad (2.15)$$

with:

$$z_{\vec{k}} = \prod_{l=1}^k k_l! l^{k_l}. \quad (2.16)$$

Therefore, we are led to

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \sum_{C(\vec{k})} \frac{1}{z_{\vec{k}}} \gamma_{\vec{k}}(U), \quad (2.17)$$

which can also be written (see e.g.

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \text{Tr}_{(k,0,\dots,0)} U, \tag{2.18}$$

as we wanted to show.

To summarize, we have shown that integrating out the degrees of freedom associated to the single separated  $D3$ -brane — when  $k$  fundamental strings are stretching between the  $D3$ -brane and a stack of  $N$   $D3$ -branes — inserts a half-BPS Wilson loop operator into the  $\mathcal{N} = 4$  SYM path integral in the  $k$ -th symmetric representation of  $U(N)$ .

We can now make contact with the  $D3_k$ -brane solution [4, 2] in  $\text{AdS}_5 \times S^5$ . The solution of the Born-Infeld equations of motion for a single  $D3$  brane with  $k$  fundamental strings stretched between that brane and a stack of  $N$   $D3$ -branes was already found in [4]. In this solution, the  $N$   $D3$ -branes are replaced by their supergravity background and the other  $D3$ -brane with the attached strings as a BION solution [15, 16]. In the near horizon limit, the  $D3$ -brane solution in [4] indeed becomes the  $D3_k$ -brane solution in  $\text{AdS}_5 \times S^5$ .

Therefore, we have given a microscopic explanation of the identification

$$D3_k \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(k,0,\dots,0)}, \tag{2.19}$$

proposed in [1].

### 3. Multiple $D3_k$ -branes as Wilson loop in arbitrary representation

In the previous section, we have shown that a single  $D3_k$ -brane corresponds to a Wilson loop in the  $k$ -th symmetric representation. We now show that an arbitrary representation  $R$  with  $P$  rows in a Young tableau can be realized by considering  $P$   $D3$ -branes.

We consider a stack of  $N + P$   $D3$ -branes and break the gauge symmetry down to  $U(N) \times U(P)$  by separating  $P$  of the branes a distance  $L$ . In the gauge theory description this corresponds to turning on a scalar expectation value as in (2.3). We also consider a background of  $k$  fundamental strings stretched between the two stacks of branes.

Therefore, we must study the low energy effective field theory of  $U(N+P)$   $\mathcal{N} = 4$  SYM when spontaneously broken to  $U(N) \times U(P)$  and in the limit  $L \rightarrow \infty$ , where the charges become infinitely massive probes.<sup>6</sup> The presence of  $k$  fundamental strings is realized in the gauge theory by inserting the creation operator of a  $k$  W-boson state at  $t \rightarrow -\infty$  and the annihilation operator of a  $k$  W-boson state at  $t \rightarrow \infty$ . The  $k$  W-boson annihilation operator is given by

$$\Psi(t) = \chi_{i_1}^{I_1}(t) \chi_{i_2}^{I_2}(t) \dots \chi_{i_k}^{I_k}(t) \tag{3.1}$$

and the  $k$  W-boson creation operator by  $\Psi^\dagger(t)$ , where<sup>7</sup>  $i_l = 1, \dots, N$  and  $I_l = 1, \dots, P$ .

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<sup>6</sup>Just as before, the  $U(P)$  gauge dynamics completely decouples from the  $U(N)$  gauge theory in the  $L \rightarrow \infty$  limit.

<sup>7</sup>The W-bosons transform in the  $(N, \bar{P})$  representation of the  $U(N) \times U(P)$  gauge group, see the appendix for details.

Such a  $k$  W-boson state transforms under  $U(N)$  and  $U(P)$  as a sum over representations with  $k$  boxes in a Young tableau. In order to project to a specific representation  $R$  we can apply to the  $k$  W-boson annihilation operator (3.1) the following projection operator

$$P_\alpha^R = \frac{d_R}{k!} \sum_{\sigma \in S_k} D_{\alpha\alpha}^R(\sigma) \sigma \tag{3.2}$$

where  $R = (n_1, n_2, \dots, n_P)$ , with  $k = \sum_i n_i$ , labels an irreducible representation of both<sup>8</sup>  $S_k$ ,  $U(N)$  and  $U(P)$ .  $D_{\alpha\beta}^R(\sigma)$  is the representation matrix for the permutation  $\sigma$  in the representation  $R$ ,  $d_R$  is the dimension of the representation  $R$  of  $S_k$  and  $\alpha, \beta = 1, \dots, d_R$ . Therefore, the operator

$$\Psi_\alpha^R(t) = P_\alpha^R \Psi = \frac{d_R}{k!} \sum_{\sigma \in S_k} D_{\alpha\alpha}^R(\sigma) \chi_{i_1}^{I_{\sigma(1)}}(t) \chi_{i_2}^{I_{\sigma(2)}}(t) \dots \chi_{i_k}^{I_{\sigma(k)}}(t) \tag{3.3}$$

describes a  $k$  W-boson state transforming in the irreducible representation  $R$  of  $S_k$ ,  $U(N)$  and  $U(P)$ .

The path integral to perform, representing our brane configuration with  $k$  fundamental strings stretching between the two stacks of  $D$ -branes, in the  $L \rightarrow \infty$  limit is given by<sup>9</sup>

$$Z = e^{iS_{N=4}} \int [D\chi][D\chi^\dagger] e^{iS_\chi} \sum_{\alpha=1}^{d_R} \Psi_\alpha^R(\infty) \Psi_\alpha^{\dagger R}(-\infty), \tag{3.4}$$

where  $S_\chi$  is the straightforward generalization of (2.6) when the gauge group is  $U(N) \times U(P)$ .

The “effective” propagator for the W-bosons is now

$$\langle \chi_i^I(\infty) \chi_j^{J\dagger}(-\infty) \rangle_{\text{eff}} \equiv \langle \exp \left( i \int dt i\chi^\dagger \partial_t \chi + \chi^\dagger (A_0 + \phi) \chi \right) \chi_i^I(\infty) \chi_j^{J\dagger}(-\infty) \rangle = U_{ij} \delta^{IJ}, \tag{3.5}$$

with  $U$  given in (2.10). The sum over all Wick contractions in (3.4) gives:

$$Z = e^{iS_{N=4}} \left( \frac{d_R}{k!} \right)^2 \sum_{\alpha=1}^{d_R} \sum_{\sigma, \tau, \omega \in S_k} D_{\alpha\alpha}^R(\sigma) D_{\alpha\alpha}^R(\tau) U_{i_{\omega(1)}}^{i_1} \dots U_{i_{\omega(k)}}^{i_k} \delta_{I_{\omega\tau(1)}}^{I_{\sigma(1)}} \dots \delta_{I_{\omega\tau(k)}}^{I_{\sigma(k)}}. \tag{3.6}$$

By appropriate change of variables, this can be simplified to

$$Z = e^{iS_{N=4}} \left( \frac{d_R}{k!} \right)^2 \sum_{\alpha=1}^{d_R} \sum_{\sigma, \tau, \omega \in S_k} D_{\alpha\alpha}^R(\sigma) D_{\alpha\alpha}^R(\tau) U_{i_{\omega(1)}}^{i_1} \dots U_{i_{\omega(k)}}^{i_k} P^{C(\sigma^{-1}\omega\tau)}, \tag{3.7}$$

where  $C(\sigma)$  is the number of disjoint cycles in the permutation  $\sigma$  and:

$$P^{C(\sigma^{-1}\omega\tau)} = \sum_{I_1, \dots, I_k} \delta_{I_{\sigma^{-1}\omega\tau(1)}}^{I_1} \dots \delta_{I_{\sigma^{-1}\omega\tau(k)}}^{I_k}. \tag{3.8}$$

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<sup>8</sup>There is a natural action of  $S_k$ ,  $U(N)$  and  $U(P)$  on  $\Psi(t)$ . The projected operator in fact transforms in the same representation  $R$  for both  $S_k$ ,  $U(N)$  and  $U(P)$  groups (see e.g. [17]). The representations of the unitary and symmetric groups are both labeled by the same Young tableau  $R = (n_1, n_2, \dots, n_P)$ .

<sup>9</sup>To avoid cluttering the formulas, the sum over  $U(N)$  and  $U(P)$  indices is not explicitly written throughout the rest of this note.

We proceed<sup>10</sup> by introducing  $\delta(\rho)$ , an element in the group algebra, which takes the value 1 when the argument is the identity permutation and 0 when the argument is any other permutation. This allows (3.7) to be written as:

$$e^{iS_{N=4}} \left(\frac{d_R}{k!}\right)^2 \sum_{\alpha=1}^{d_R} \sum_{\sigma, \tau, \omega, \rho \in S_k} D_{\alpha\alpha}^R(\sigma) D_{\alpha\alpha}^R(\tau) U_{i_{\omega(1)}}^{i_1} \dots U_{i_{\omega(k)}}^{i_k} P^{C(\rho)} \delta(\rho^{-1} \sigma^{-1} \omega \tau). \quad (3.9)$$

Summing over  $\tau$  yields

$$e^{iS_{N=4}} \left(\frac{d_R}{k!}\right)^2 \sum_{\alpha=1}^{d_R} \sum_{\sigma, \omega \in S_k} D_{\alpha\alpha}^R(\sigma) D_{\alpha\alpha}^R(\omega^{-1} \sigma \sum_{\rho \in S_k} \rho P^{C(\rho)}) U_{i_{\omega(1)}}^{i_1} \dots U_{i_{\omega(k)}}^{i_k}. \quad (3.10)$$

Since  $\mathcal{C} = \sum_{\rho \in S_k} \rho P^{C(\rho)}$  commutes with all elements in the group algebra, we can use the identity  $D_{\alpha\alpha}^R(\mathcal{C}\sigma) = \frac{1}{d_R} D_{\alpha\alpha}^R(\sigma) \chi_R(\mathcal{C})$ , where  $\chi_R(\mathcal{C}) = \sum_{\alpha=1}^{d_R} D_{\alpha\alpha}^R(\mathcal{C})$  is the character of  $S_k$  in the representation  $R$  for  $\mathcal{C}$ . Therefore, (3.10) reduces to

$$e^{iS_{N=4}} \frac{d_R}{k!} \text{Dim}_P(R) \sum_{\alpha=1}^{d_R} \sum_{\sigma, \omega \in S_k} D_{\alpha\alpha}^R(\sigma) D_{\alpha\alpha}^R(\omega^{-1} \sigma) U_{i_{\omega(1)}}^{i_1} \dots U_{i_{\omega(k)}}^{i_k}, \quad (3.11)$$

where

$$\text{Dim}_P(R) = \frac{1}{k!} \sum_{\sigma \in S_k} \chi_R(\sigma) P^{C(\sigma)} \quad (3.12)$$

is the dimension of the irreducible representation  $R$  of  $U(P)$ . By using the relation satisfied by the fusion of representation matrices

$$\sum_{\sigma \in S_k} D_{\alpha\alpha}^R(\sigma) D_{\alpha\alpha}^R(\omega^{-1} \sigma) = \frac{k!}{d_R} D_{\alpha\alpha}^R(\omega^{-1}) \quad (3.13)$$

we arrive at:

$$Z = e^{iS_{N=4}} \text{Dim}_P(R) \sum_{\omega \in S_k} \chi_R(\omega) U_{i_{\omega(1)}}^{i_1} \dots U_{i_{\omega(k)}}^{i_k}. \quad (3.14)$$

Finally, we use the Frobenius character formula (see e.g. [14]), which relates the trace of a matrix  $U$  in an arbitrary representation  $R = (n_1, n_2, \dots, n_P)$  of  $U(N)$  to the trace in the fundamental representation

$$\text{Tr}_R(U) = \frac{1}{k!} \sum_{\omega \in S_k} \sum_{i_1, \dots, i_k} \chi_R(\omega) U_{i_{\omega(1)}}^{i_1} \dots U_{i_{\omega(k)}}^{i_k} \quad (3.15)$$

to show that the final result of the path integral is

$$Z = e^{iS_{N=4}} \cdot k! \text{Dim}_R(M) \text{Tr}_R(U), \quad (3.16)$$

the insertion of a half-BPS Wilson loop in the representation  $R$ .

In the near horizon limit, when the  $N$  D3-branes are replaced by their near horizon geometry, the  $P$  D3-branes with the array of stretched fundamental strings labeled by

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<sup>10</sup>The paper [18] has a useful compilation of useful formulas relevant for this paper.



$R = (n_1, n_2, \dots, n_P)$  become the brane configuration  $(D3_{n_1}, D3_{n_2}, \dots, D3_{n_P})$  in  $\text{AdS}_5 \times \text{S}^5$ , thus arriving at the identification<sup>11</sup>

$$(D3_{n_1}, \dots, D3_{n_P}) \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(n_1, \dots, n_P, 0, \dots, 0)} \quad (3.17)$$

in [1].

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### A. Gauge theory along Coulomb branch

The low energy dynamics of a stack of  $N + P$  coincident  $D3$ -branes is described by four dimensional  $\mathcal{N} = 4$  SYM with  $U(N + P)$  gauge group. The spectrum of the theory includes a vector field  $\hat{A}_\mu$ , six scalar fields  $\hat{\Phi}_i$  and a ten dimensional Majorana-Weyl spinor  $\hat{\Psi}$ . The action is given by

$$\hat{S}_{\mathcal{N}=4} = \frac{1}{2g_{\text{YM}}^2} \int \text{Tr} \left( -\frac{1}{2} \hat{F}_{\mu\nu}^2 - (\hat{D}_\mu \hat{\Phi}_i)^2 + \frac{1}{2} [\hat{\Phi}_i, \hat{\Phi}_j]^2 - i \hat{\Psi} \Gamma^\mu \hat{D}_\mu \hat{\Psi} - \hat{\Psi} \Gamma^i [\hat{\Phi}_i, \hat{\Psi}] \right), \quad (\text{A.1})$$

where each field is in the adjoint of the gauge group  $U(N + P)$ . We use real ten dimensional gamma matrices  $\Gamma^i$  and  $\Gamma^\mu$  and we choose  $\Gamma^0$  as charge conjugation matrix. Thus, the Majorana-Weyl spinor  $\lambda$  has 16 real components and  $\bar{\lambda} = \lambda^T \Gamma^0$ .

Now we separate a stack of  $P$  branes from the remaining stack of  $N$  branes, i.e. we give a non trivial vacuum expectation value to the scalar fields. Without loss of generality, we take

$$\langle \hat{\Phi}_9 \rangle = \begin{pmatrix} 0 & 0 \\ 0 & LI_P \end{pmatrix}, \quad (\text{A.2})$$

where  $I_P$  is the  $P \times P$  unit matrix and  $L$  is a constant with the dimensions of mass. To expand the action around this vacuum, we first define the fields as

$$\hat{A}_\mu = \begin{pmatrix} A_\mu & \omega_\mu \\ \omega_\mu^\dagger & \tilde{A}_\mu \end{pmatrix} \quad \hat{\Phi}_i = \begin{pmatrix} \Phi_i & \omega_i \\ \omega_i^\dagger & \delta_{i9} LI_P + \tilde{\Phi}_i \end{pmatrix} \quad \hat{\Psi} = \begin{pmatrix} \Psi & \theta \\ \theta^\dagger & \tilde{\Psi} \end{pmatrix}, \quad (\text{A.3})$$

where  $A_\mu$ ,  $\Phi_i$  and  $\Psi$  transform in the adjoint representation of  $U(N)$  and  $\tilde{A}_\mu$ ,  $\tilde{\Phi}_i$  and  $\tilde{\Psi}$  transform in the adjoint representation of  $U(P)$ .  $\omega_\mu$ ,  $\omega_i$  and  $\theta$  are W-bosons fields and transform in the  $(N, \bar{P})$  representation of the gauge group  $U(N) \times U(P)$ .

The action becomes:

$$\hat{S}_{\mathcal{N}=4}^L = S_{\mathcal{N}=4} + \tilde{S}_{\mathcal{N}=4} + S_W + S_{\text{interactions}}. \quad (\text{A.4})$$

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<sup>11</sup>We can trivially reabsorb the overall constant in (3.16) in the normalization of  $\Psi$ .

$S_{\mathcal{N}=4}$  and  $\tilde{S}_{\mathcal{N}=4}$  are the actions for the effective field theories living on the two stacks of branes, i.e. four dimensional  $\mathcal{N} = 4$  SYM with gauge group respectively  $U(N)$  and  $U(P)$ .  $S_W$  is the action for the W-bosons and their superpartners

$$S_W = \int \text{Tr} \left( -\frac{1}{2} f_{\mu\nu}^\dagger f^{\mu\nu} - L^2 \omega_\mu^\dagger \omega^\mu - \sum_{i=4}^9 \partial^\mu \omega_i^\dagger \partial_\mu \omega_i - L^2 \sum_{k=4}^8 \omega_k^\dagger \omega_k - i \bar{\theta}^\dagger \Gamma^\mu \partial_\mu \theta + L \bar{\theta}^\dagger \Gamma^9 \theta + \dots \right), \quad (\text{A.5})$$

where  $f_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$  and  $\dots$  denote terms fourth order in the W-boson fields.  $S_{\text{interactions}}$  is the action describing the interactions between the W-bosons and the fields living on the two stacks of branes. It includes terms of the third and fourth order in the fields.

We are interested in the limit where the two stacks of branes are infinitely separated, i.e. in the limit where  $L \rightarrow \infty$ . From the quadratic action (A.5) we see that the W-bosons  $\omega_m$  with  $m = 1, \dots, 8$  and the fermions become infinitely massive. Taking the infinite mass limit of a relativistic massive field, corresponds to considering the non-relativistic limit. The surviving dynamics in the limit can be explicitly extracted by making the following redefinition:

$$\omega_m = \frac{1}{\sqrt{L}} e^{-itL} \chi_m \quad \text{where} \quad m = 1, \dots, 8. \quad (\text{A.6})$$

For the W-bosons superpartners, which also become infinitely massive, we first define

$$\Gamma_{09} \theta_\pm = \pm \theta_\pm, \quad (\text{A.7})$$

where this projection must be understood in the spinors space. To extract the physics in the limit we make the following rescaling:

$$\theta = \theta_+ + \theta_- = e^{-iLt} \xi_+ + e^{-iLt} \xi_-. \quad (\text{A.8})$$

Considering (A.6) and (A.8) and then taking the infinite mass limit  $L \rightarrow \infty$  the W-boson action (A.5) reduces to

$$S_W^{\text{NR}} = \int \text{Tr} \left( i \sum_{m=1}^8 \chi_m^\dagger \partial_t \chi_m + i (\xi_+^T)^\dagger \partial_t \xi_+ \right) \quad (\text{A.9})$$

where the transposition is in the space of fermions and the hermitian conjugation is in the matrix space. The  $\xi_-$  fermions become infinitely massive and decouple from the theory, as expected, since there are no antiparticles in the non-relativistic limit.

The interaction action  $S_{\text{interaction}}$  in (A.4) is now given by

$$S_{\text{interactions}}^{\text{NR}} = \int \sum_{m=1}^8 \text{Tr} \left( \chi_m^\dagger (A_0 + \Phi_9) \chi_m - \chi_m^\dagger \chi_m (\tilde{A}_0 + \tilde{\Phi}_9) + \xi_m^\dagger (A_0 + \Phi_9) \xi_m - \xi_m^\dagger \xi_m (\tilde{A}_0 + \tilde{\Phi}_9) \right) \quad (\text{A.10})$$

where  $\xi_m$  ( $m = 1, \dots, 8$ ) are the spinor components of  $\xi_+$ . All higher order terms in (A.4) vanish in the  $L \rightarrow \infty$  limit. Note that, in this limit the dynamics of the  $U(P)$  gauge theory effectively decouples from the  $U(N)$  gauge theory.

Therefore, we can then write the action describing the coupling of the W-bosons to  $U(N)$   $\mathcal{N} = 4$  SYM as<sup>12</sup>

$$S = S_{\mathcal{N}=4} + \sum_{m=1}^8 S_m, \quad (\text{A.11})$$

where  $S_{\mathcal{N}=4}$  is the action of  $\mathcal{N} = 4$  SYM with gauge group  $U(N)$  while  $S_m$  with  $m = 1, \dots, 8$  is the action for one of the eight non-relativistic supersymmetric W-bosons

$$S_m = \int [(\chi_m^\dagger)_i^I \Delta_{ij}^{IJ} (\chi_m)_j^J + (\xi_m^\dagger)_i^I \Delta_{ij}^{IJ} (\xi_m)_j^J], \quad (\text{A.12})$$

where

$$\Delta_{ij}^{IJ} = i\delta_{ij}\delta^{IJ}\partial_t + (A_0 + \Phi_9)_{ij}\delta^{IJ}, \quad (\text{A.13})$$

which is what we have used in the main text.

We note that integrating out the degrees of freedom associated to the W-bosons, without any insertions, we get

$$\begin{aligned} Z &= \int \prod_{i=1}^8 ([D\chi_m][D\chi_m^\dagger][D\xi_m][D\xi_m^\dagger]) e^{iS} \\ &= e^{iS_{\mathcal{N}=4}} \frac{(\det\Delta)^{n_F}}{(\det\Delta)^{n_B}} = e^{iS_{\mathcal{N}=4}}, \end{aligned} \quad (\text{A.14})$$

where in the last step we used that  $n_F = n_B$ . Note that we recover the expected result that the metric in the Coulomb branch of  $\mathcal{N} = 4$  gets no corrections upon integrating out the massive modes.

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<sup>12</sup>There is decoupled contribution for the  $U(P)$  gauge theory which does not talk to  $U(N)$  SYM.

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